Problem Set 2

Problem 1. Dynamic Programming Consider the following planning problem in an economy with habit formation:

$$\max \sum_{t=0}^{\infty} \beta^t \log \left(c_t - \alpha c_{t-1}\right)$$

subject to

$$\begin{aligned} c_t + k_{t+1} &\leq Ak_t \\ c_t, k_{t+1} &\geq 0, \forall t \geq 0 \\ k_0, c_{-1} &: \text{ given} \end{aligned}$$

The preferences here are said to exhibit habit persistence since high consumption in period $t-1$ increases marginal utility in period t and thus households want to keep their consumption high at t.

a. Write this problem in canonical form.

b. Write the Bellman equation associated with the above canonical form. Can you use the standard tools from Stokey Lucas to prove that the solution to the functional equation exists? If no, why not?

c. Show that the value function associated with the above problem is homogeneous of degree 1 and as a result it can be simplified to having only one state variable.

d. If your answer to the second part of question b is no, use the formulation in part c to show existence of unique solution to the Bellman equation.

e. Characterize the Balanced Growth Path or steady state of this economy in the long-run. Show that the economy converges to a Balanced Growth Path.

f. Describe how the model behaves as it transitions to BGP.

Problem 2. Recursive Preferences In most of our problems we have assumed that preferences are additively separable over time. Here you extend the analysis to economies where this is not the case. To do so, we define the following class of recursive preferences: For a sequence of consumption $\mathbf{c} = \{c_t\}_{t=0}^{\infty}$, utility is defined via the following recursive formulation:

$$W(\mathbf{c}) = G(c_0, W(\mathbf{c}_1))$$

where $\mathbf{c}_1 = \{c_t\}_{t=1}^{\infty}$ and G is a continuous function that is increasing in its arguments. Suppose that capital accumulation and production are the same as the standard one-sector growth model.

a. Write the planning problem for this economy assuming that the representative agent has the above recursive preferences and its associated Bellman equation.

b. Provide sufficient condition on G so that you can apply the techniques in Stokey Lucas and show that the solution to the Bellman equation is unique. Provide some examples of G that satisfy such conditions

c. Using the class of G's that you came up with in part b, describe the steady state of the economy. *Hint: Assume differentiability of the value function and use the Envelope Theorem.*

Problem 3. Balanced Growth Path and Investment Specific Technical Progress One observation that we have ignored in our models of economic growth is the observation that price of capital (equiment and software) has decreased persistently over time. In this problem you will investigate how to augment our model in order to capture this observation.

One way to capture this observation is to allow investment to become more productive - this is literally equivalent to having price of investment fall over time. In other words, suppose that feability and law of motion of capital are now given by

$$c_t + x_t = zF\left(k_t, \overline{e}\right)$$
$$k_{t+1} = k_t\left(1-\delta\right) + x_t q_t$$

where

$$\frac{q_{t+1}}{q_t} = 1 + g_q$$

and $F\left(k, n\right) = k^{\alpha} n^{1-\alpha}$. Note that there is no population growth and other forms of technological change.

One can think of $1/q_t$ as the price of investment relative to consumption and given the above formulation, it is as if investment is becoming cheaper over time.

a. Formulate and define a competitive equilibrium for this economy.

b. Suppose that a BGP exists. Calculate the growth rate of the economy on a BGP.

c. Use the solution to the above problem to write the planning problem associated with the competitive equilibrium recursively. Show that a unique BGP exists and that the economy converges to BGP in the long-run.

d. Calculate the rental price of capital/equipment in this economy. Do the same for the real interest rate.

e. Now suppose that in addition to equipment, there is also structures. In particular, suppose that production function is given by

$$F\left(k_t^e, k_t^s, \overline{e}\right) = \left(k_t^s\right)^{\alpha_e} \left(k_t^s\right)^{\alpha_s} \overline{e}^{1-\alpha_e-\alpha_s}$$

where the law of motion of structures and equipment are given by

$$k_{t+1}^s = k_t^s\left(1-\delta_s\right) + i_t$$
$$k_{t+1}^e = k_t^e\left(1-\delta_s\right) + q_t x_t$$

while feasibility is given by

$$c_t + i_t + x_t = z_t F\left(k_t^e, k_t^s, \bar{e}\right).$$

with

$$\frac{z_{t+1}}{z_t} = 1 + g_z, \frac{q_{t+1}}{q_t} = 1 + g_q$$

Calculate the growth rate of the BGP in this model. What happens to the ratio of equipment to structures in the long-run? What happens to the rate of return on equipment as well as the real interest rate?

Problem 4. Structural Transformation and Long-run Dynamics Consider an extension of the model in class whith two consumption goods 1 and 2. Suppose there is no population growth and a representative household exists with a utility function of the form

$$\sum_{t=0}^{\infty} \beta^t \left[a \log c_{1,t} + (1-a) \log c_{2,t}\right]$$

where in the above $c_{1,t}$ and $c_{2,t}$ are consumption of goods 1 and 2. Furthermore, $a < 1$. Suppose that good 1 can be interchangeably used for consumption and investment while good 2 can only be consumed. The production of the two goods are given by

$$y_{1t} = k_{1t}^{\alpha}\left(A_{1t} n_{1t}\right)^{1-\alpha}, y_{2t} = k_{2t}^{\alpha}\left(A_{2t} n_{2t}\right)^{1-\alpha}$$

where k_{jt}, n_{jt} are capital and labor employed in production of each good while A_{jt} is the labor-augmenting productivity in each sector. Suppose that $\frac{A_{jt+1}}{A_{jt}} = 1 + g_j > 1$. Assume that capital depreciates at rate δ while total labor is normalized to 1.

a. Define a competitive equilibrium for this economy using the time-0 trading assumption.

b. Write down the pareto optimal problem associated with the economy above.

c. Suppose that $g_1 = g_2$. Show that you can write the problem recursively. Define a Balanaced Growth Path for this economy and calculate its long-run growth rate.

d. Now suppose that $g_1 < g_2$. What can you say about the behavior of this economy in the long-run and its growth rate on a balanced growth path.

Problem 5. Solve exercise 15.3, 15.4, 15.5 in Ljunqgvist and Sargent.

Problem 6. Growth and Inequality Consider an economy that is consisted of two individuals that are only different in terms of their time-0 endowment of capital. The rich and the poor! They supply one unit of labor inelastically and their utility functions are given by

$$\sum_{t=0}^{\infty} \beta^t \log\left(c_t^i - \bar{c}\right)$$

Suppose further that consumption and investment goods are produced using the same production function which is a Cobb-Douglas production function $F(k, n) = Ak^{\alpha}n^{1-\alpha}$. As we have shown in class any competitive equilibrium of this economy is pareto optimal and we have provided the formulation of the programming problem associated with that.

a. For a given set of welfare weights (α_R, α_P) with $\alpha_R + \alpha_P = 1$, formulate the pareto optimal problem as recursive problem with aggregate capital as the only state variable.

b. Calculate the steady state value of capital stock for this economy for a given set of welfare weights.

c. How does inequality, as measured by the ratio of consumption between the rich and the poor evolve over time as the economy converges to its steady state?

d. Use the notion of wealth that you defined in problem 4 of PS1 to calculate the wealth of each individual. How does wealth inequality change over time as the economy converges to steady state?

Problem 7. Time-Inconsistent Preferences Consider the following sequence problem:

$$\max_{c_t, k_{t+1}} \log c_0 + \delta \sum_{t=0}^{\infty} \beta^t \log c_t$$

subject to

$$\begin{aligned} c_t + k_{t+1} &\leq Rk_t, \\ k_0 &: \text{ given} \end{aligned}$$

where $\delta < \beta < 1$.

a. Try to solve for consumption using lagrangian methods. Can we write this problem as a dynamic programming problem?

b. Suppose each household is comprised of infinitely-many selves. A "t-self" has preferences given by

$$\log c_t + \delta \sum_{s=t+1}^{\infty} \beta^{s-t} \log c_s$$

Show that if we consider the solution to the sequence problem above for 0-self, given k_1 utility of 1-self is not maximized.

c. Now assume that each self takes into account the action of other selves. That is if a t-self starts with capital k, he will save $g(k)$ and a $t-1$-self takes this into account. Try to write this new problem in a recursive way.
Hint: Do not try to write a Bellman equation!

d. Find a solution to the recursive formulation above using guess and verify. Note that this solution won't be unique.